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Invariants of Topological Quantum Mechanics

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We investigate a new topological invariant of the punctured plane using a Hamiltonian approach. The Hamiltonian is built out of topological invariants available on the punctured plane. On the other hand it is shown that the model is a generalized version, using the appropriate language of homotopy, of the superconformal quantum mechanics (gauge approach) recently proposed by L. Baulieu *et al.* This relationship allows a better understanding of the structure and results of the gauge approach and makes possible a proper identification of the topological invariants which emerge from it.

1. INTRODUCTION

Baulieu and Rabinovici (1993) investigated superconformal quantum mechanics. The punctured plane was chosen as the target space on which closed trajectories around the hole were selected. The motivation of the authors behind the study of loops in the punctured plane was to understand the mechanism of supersymmetry breaking, which could provide nonvanishing mean values to topological observables which are BRST-exact, and this is an open question of current interest (Myers, 1990). The system considered has a nontrivial but simple topology, characterized by the winding number of the loops. The model is thus a supersymmetric quantum mechanical system defined on the punctured plane. Like any supersymmetric quantum mechanical action, it can be regarded as a topological action (Witten, 1988a–c; see Birmingham *et al.*, 1991, for review) by a Wick rotation. The use of local BRST invariance (Baulieu and Arago de Carvalho, 1991a,b; Birmingham *et al.*, 1990; Birmingham and Rakowski, 1989; Delduc *et al.*, 1989) input, although not a physical principle, allows the authors to select a superconformal potential which turns out to be solvable and possess interesting properties. The striking feature of the spectrum obtained is that it has no admissible

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normalizable ground state ($E = 0$). The supersymmetry is thus broken without the presence of a dimensioned parameter and this opens the possibility to have nonvanishing mean values of BRST-exact observables. Note that this type of supersymmetric breaking is shown not to be true at one spatial dimension and requires a special choice of the potential in two dimensions. However, the way the invariants emerged from the formalism makes their identification ambiguous, as they are written in terms of BRST charges and ghost fields. A first but not very satisfying attempt was made by Baulieu and Rabinovici (1993) to connect some of the invariants to the superconformal generators.

The aim of the present paper is first to identify the topological invariants of the punctured plane, a subject of particular interest in itself. The procedure is an intrinsic and direct study of their existence using only topological information available on the punctured plane. The model we propose is not a gauge theory, but a quantum mechanical system whose Hamiltonian is built out of topological operators only (Hu, 1959; Maunder, 1979; Balachandran *et al.*, 1991). These operators are the winding number and the fundamental group of homotopy. The model is solvable and is in fact, as we shall see, another generalized version of the gauge field model, but interpreted in the language of homotopy in which gauge transformations correspond to homotopic deformations. It allows, in addition to a proper identification of observables in the gauge model, the possibility of making certain results of the gauge approach more plausible, such as the number of well-defined invariants (there are too many ill-defined invariants), the necessary dimension two and not one of the target space (Fubini and Rabinovici, 1984; de Alfaro *et al.*, 1976) and the special choice of the coupling constant of the superconformal potential.

The rest of the paper is organized as follows. First we summarize the main results of Baulieu and Rabinovici (1993). Then we describe our model in detail, and finally we conclude with a comparative study in which we emphasize the features common to both models and end up with the identification of the observables.

2. THE GAUGE FIELD APPROACH

The topological classical action of interest is the closed-form integral

$$S_{cl}[\mathbf{q}] = \beta \oint d\theta \tag{1}$$

$$= \oint d\tau \, \beta \, \frac{\epsilon^{ij} \dot{q}_i q_j}{\mathbf{q}^2} \tag{2}$$

where β is a real coupling constant, the q_i are the Cartesian coordinates, and θ is the polar coordinate. This action is a measure of the winding number of the corresponding loop up to the constant β. The Hamiltonian associated to this action is zero, however, as the corresponding Lagrangian is first order in $\dot{\mathbf{q}}$. The above action, on the other hand, is invariant under the gauge symmetry

$$\mathbf{q}(t) \to \mathbf{q}(t) + \boldsymbol{\epsilon}(t) \tag{3}$$

where $\boldsymbol{\epsilon}(t)$ is any local shift of the particle position $\mathbf{q}(t)$ which does not change the winding number of the trajectory. To gauge fix this action with a quadratic dependence on the velocity $\dot{\mathbf{q}}$ one chooses a gauge function of the form $\dot{q}_i + \delta V/\delta q_i$, where the potential is an arbitrary function. To get interesting topological information one has recourse to local BRST symmetry, which selects the (super)conformal potential of the form

$$\frac{\beta^2}{2\mathbf{q}^2} \tag{4}$$

As an illustration we give the Hamiltonian thus obtained:

$$\begin{aligned} H &= \frac{1}{2}\{Q, \overline{Q}\} \\ Q &= -i\psi_r \frac{\partial}{\partial r} - i\frac{\psi_\theta}{r}\left(\frac{\partial}{\partial\theta} - \beta\right) \end{aligned} \tag{5}$$

where the fields $\boldsymbol{\psi}(t)$ are the topological ghosts associated with the particles positions $\mathbf{q}(t)$ and Q is the BRST charge. The eigenstates of the Hamiltonian are labeled by their nonnegative energies E, angular momentum n, and the ghost number α, with $\alpha = 1, 2, 3, 4$, and are denoted $|E, n, \alpha\rangle$.

The topological invariants in this model are defined in terms of BRST charges and ghost (antighost) fields. The candidates for such invariants are (from dimensional arguments for the first two)

$$\begin{aligned} \{Q, r\overline{\psi}_\theta\} &= \{\overline{Q}, r\psi_\theta\}^\dagger \\ \{Q, r\overline{\psi}_r\} &= \{\overline{Q}, r\psi_r\}^\dagger \\ \Delta &= \mathrm{Tr}(-)^F \exp(-\gamma H) \end{aligned} \tag{6}$$

where $(-)^F$ is the ghost or fermion operator, Tr means trace over E, n, and γ is a constant. The mean value of the first operator between any normalized state $|\phi_n\rangle = \int dE\, \rho(E)|E, n\rangle$ is, independent of the weighting function $\rho(E)$,

$$\langle\phi_n|\{Q, r\bar{\psi}_\theta\}|\phi_n\rangle = n + i\beta \tag{7}$$

The second and the third (presumed) invariants in equation (6) turn out to be ill defined, the second was rejected by the above authors, and the third one, the Witten index (Witten, 1982; Alvarez Gaumé, 1983; Friedan and Windey, 1984; Alvez *et al.*, 1985; Akhoury and Comtet, 1984), is somehow "regularized." The latter "observable" has no analog in our approach, which incorporates only even objects. The Witten index counts by definition the relative number of bosonic and fermionic zero-energy modes. The gauge approach therefore contains effectively only one well-defined topological invariant.

3. THE HOMOTOPY GROUP APPROACH

The aim of our investigation is to work out the possible topological invariants of the punctured plane starting from well-defined invariants such as the winding number and the homotopy group operators. Our strategy is to look at loops on the plane as intrinsic "physical" objects subject to interactions with an external background field and having well-defined self-interactions as well. The external field is assumed to couple directly to the winding charges. Let $|n\rangle$ be the state of the loop with winding charge n. Denote by W the winding number operator and $\Pi(n)$, $n \in Z$, an element of the homotopy group of the punctured plane which is isomorphic to the set of integer Z. We then have the defining relations

$$\begin{aligned} W|n\rangle &= n|n\rangle \\ \Pi(n)|m\rangle &= e^{im\zeta}|n + m\rangle \\ \langle n|m\rangle &= \delta_{nm} \\ \sum_{n\in Z} |n\rangle\langle n| &= 1 \end{aligned} \tag{8}$$

In the following we set $\zeta = 0$ without loss of generality. Let us first consider the interaction of loops with the external field which couples to the winding charges in a more general way as

$$H_0 = g(W) \tag{9}$$

The interaction with the external field makes the $|n\rangle$ states evolve in time in such a way that they may lose their loop structure. This is made possible through the extra phase factor $\exp[-ig(n)t]$ they get from the Hamiltonian in equation (9). The necessary and sufficient condition for the loops to keep their structure with time (i.e., in the presence of the background field) is that the action of the Π operators on the evolved states remains unchanged up to a phase,

$$\begin{aligned}\Pi(m)|n, t\rangle &= \exp[-ig(n)t]\, \Pi(m)|n\rangle \\ &= \exp\{-i[g(n) - g(n+m)]t\}\, |n+m, t\rangle \end{aligned} \tag{10}$$

For this to happen we require that the function g be linear, i.e., $g(n) = \beta n$,

$$\Pi(m)|n, t\rangle = \exp(i\beta m t)\, |n+m, t\rangle \tag{11}$$

The external field is then assumed to couple linearly to the winding charge in order not to affect the loop structure. A quadratic coupling βW^2, for instance, would be insensitive to oriented loops and hence get missed. We thus have

$$H_0 = \beta W \tag{12}$$

The real number β represents the strength of the field coupling. Now the most general interaction terms one may add to the Hamiltonian H_0 and which describe loop self-interactions in the presence of the background field would have the form

$$\sum_{n,m \in Z, n \neq 0, m \neq 0} \lambda(\beta, n, m) W^m \Pi(n) \tag{13}$$

Note that higher powers of the Π are already included using the group property $\Pi^n(m) = \Pi(nm)$ and also that the commutation relations $[W, \Pi(m)] = m\Pi(m)$ which we may deduce from equations (8) have been used to move all W's to the left of the Π's. Recall that our main concern here is to look for new topological invariants which live on the punctured plane. The general interaction introduced above may lead, in the manner of nonlinear couplings to the external field, to states having nothing to do with loops. In order to capture possible intrinsic topological configurations as a result of the above interactions we must select the appropriate ones. For this purpose we only keep the Π terms [$m = 0$ in equation (13)] and factorize the λ function as follows: $\lambda(\beta, n) = \beta\lambda(n)$. This coupling function is further subject to the Hermiticity constraint $\lambda^*(-n) = \lambda(n)$. The final selected Hamiltonian which incorporates loop self-interactions in the presence of an external background without breaking the loop structure has the very simple form

$$\begin{aligned} H(\beta, \lambda) &= \beta W_\lambda \\ W_\lambda &= W + \sum_{n \in Z} \lambda(n)\Pi(n) \end{aligned} \tag{14}$$

The form of the selected Hamiltonian is analogous to that of the Hamiltonian H_0 which describes the coupling of the external field to loops without self-interactions. We therefore see that in the presence of loop self-interactions the external field couples in fact to new charges: effective charges whose values are not necessarily integers. We will call this new observable the effective winding number and denote it W_λ.

What kind of states are the eigenstates of this new object. First we solve for the eigenvalue problem. The solutions are exact (not perturbative) and are of the form[2]

$$|n_\lambda\rangle = \exp\left[-\sum_{m\in Z/\{0\}} \frac{\lambda(m)}{m}\,\Pi(m)\right]|n\rangle \tag{15}$$

For the series above to converge a judicious choice of the function λ is necessary subject to the Hermiticity requirement. One can also compute the eigenvalues of the effective winding number operator and show that they are related to the integer winding numbers through a simple relation in which only the values of the coupling function at $m = 0$ contributes to the spectrum:

$$n_\lambda = n + \lambda(0) \tag{16}$$

It is not useful in the present analysis to keep an m-dependent function λ, we but restrict ourselves to the simplest convergent case, $\lambda(m) = \lambda$. This reduces the expression of the states $|n_\lambda\rangle$ to the compact and suggestive form

$$\begin{aligned} |n_\lambda\rangle &= e^{-iJ\lambda}|n\rangle \\ J &= -i \sum_{m\in Z/\{0\}} \frac{\Pi(m)}{m} \end{aligned} \tag{17}$$

The fact that the new states $|n_\lambda\rangle$ are the unitary transform of the old ones (J is Hermitian) and that the spectrum undergoes just a constant shift means that the physics is unaffected by the introduction of the above specific loop self-interaction. It is merely a change of basis.

Before proceeding to show the formal resemblance of both approaches and identify the invariants, we want to be complete and show that the newly defined states possess interesting properties. First, they reduce to old states $|n\rangle$ up to a phase when the coupling λ takes integer values,

$$|n_\lambda\rangle_{\lambda=m} = (-1)^m|n + m\rangle \tag{18}$$

Second, although they are not eigenstates of the winding number operator, the state $|n_\lambda\rangle$ still carries the winding charge n on average, that is,

[2]These states together with the defining equation (15) are fundamental, as they constitute topological building blocks of Bessel functions (Mekhfi, n.d.).

$$\langle n_\lambda | W | n_\lambda \rangle = n \tag{19}$$

Third, these states are a basis for the representation space of the W_λ and Π operators, that is,

$$\begin{aligned} W_\lambda | n_\lambda \rangle &= (n + \lambda) | n_\lambda \rangle \\ \Pi(m) | n_\lambda \rangle &= | (n + m)_\lambda \rangle \end{aligned} \tag{20}$$

It is the second equation in (20) which is in fact behind the choice of the type of interactions of loops we adopted, namely the interaction does not destroy the loop structure. The second and the third properties are straightforward. To see how the first property occurs, we convert to the one-dimensional irreducible representations of the fundamental group $\Pi \sim Z$. We denote these representation states $|\alpha\rangle$ and define them as follows:

$$\Pi(m) | \alpha \rangle = e^{-im\alpha} | \alpha \rangle, \qquad 0 < \alpha \leq 2\pi \tag{21}$$

The representations $|n\rangle$ and $|\alpha\rangle$ are related to each other through the Fourier transform

$$|n\rangle = \int_0^{2\pi} |\alpha\rangle e^{-in\alpha} \frac{d\alpha}{2\pi} \tag{22}$$

Using the above formula, we can write the new states in the integral form

$$\begin{aligned} |n_\lambda\rangle &= e^{-iJ\lambda} \int_0^{2\pi} |\alpha\rangle e^{-in\alpha} \frac{d\alpha}{2\pi} \\ &= \int_0^{2\pi} |\alpha\rangle \exp\left(-\lambda \sum_{m \in Z/\{0\}} \frac{e^{-im\alpha}}{m} \right) e^{-in\alpha} \frac{d\alpha}{2\pi} \\ &= e^{-i\lambda\pi} \int_0^{2\pi} e^{-i(n+\lambda)\alpha} \frac{d\alpha}{2\pi} |\alpha\rangle \end{aligned} \tag{23}$$

To perform the last step we made use of the result

$$\sum_{m \in Z/\{0\}} \frac{e^{-im\alpha}}{m} = i(\alpha - \pi) \tag{24}$$

Now that we have defined the effective winding number operator W_λ as the new topological invariant of the punctured plane and studied its spectrum and the properties of its associated eigenvectors, it is appropriate to make a comparative study of both approaches and ultimately identify W_λ with the unique well-defined topological invariant of the gauge model. For this we need to compute the mean value of our invariant on the $|n\rangle$ basis, as the states $|n_\lambda\rangle$ do not enter the analysis of the other approach. We have

$$\begin{aligned}\langle n | W_\lambda | n\rangle &= \langle n | W | n\rangle + \lambda \sum_{m \in Z} \langle n | \Pi(m) | n\rangle \\ &= n + \lambda \end{aligned} \tag{25}$$

4. A COMPARATIVE STUDY

As we briefly sketched in Section 2, one starts in the gauge model with a topological action which is a measure of the winding number of the closed configuration times $2\pi\beta$, with β being a coupling constant. In the homotopy group approach one starts with a Hamiltonian which describes the (linear) coupling of an external field to the winding charges. The coupling is chosen on the basis that the field does not destroy the loop structure of the system,

$$\beta \int d\theta \Leftrightarrow \beta W \tag{26}$$

The topological action in equation (1), written in terms of the coordinates q_i [see equation (2)], exhibits a gauge symmetry invariance, but as it stands it exhibits rather invariance under the homotopy group deformations, that is, the n-preserving deformations of closed configurations. We thus have

$$\mathbf{q} \to \mathbf{q} + \boldsymbol{\epsilon} \Leftrightarrow \text{Homotopic deformation} \tag{27}$$

Gauge fixing a symmetry breaks the latter in such a way that the physics is not affected. In the process of gauge fixing topological theories, different potentials all equally possible may be introduced. In the above gauge model local BRST input is necessary in order to get topological invariants. It allows us to select a (super) conformal potential. The coupling constant of the associated potential turns out to be necessarily the square of the coupling constant β already introduced into the unfixed action. Only in this case do well-defined topological invariants emerge. In the homotopy group model, an interaction of a special type has been introduced into the Hamiltonian, the selection criterion being that the introduced potential does not affect the loop structure. This requirement has the important consequence that it leads to a mere change of basis. This change of basis did not affect the physics [see equations (16) and (17)] and this is what mimics the gauge fixing in the gauge approach. As for the gauge model, in order to capture new topological invariants, the coupling constant should necessarily be proportional to the external field strength β. This last requirement is the same for both models, except that the proportionality constant λ is here arbitrary and not necessarily equal to β. We thus have the set of similarities

$$\text{No ground state} \Leftrightarrow \text{No ground state}$$

$$|\phi_n\rangle \Leftrightarrow |n\rangle$$

$$\text{No room} \Leftrightarrow |n_\lambda\rangle$$

$$\text{Interactions fix the gauge} \Leftrightarrow \text{Interactions change the basis}$$

$$\frac{\beta^2}{\mathbf{q}^2} \Leftrightarrow \beta\lambda\Pi(0)$$

$$\text{No room} \Leftrightarrow \beta\lambda \sum_{m \in Z/\{0\}} \Pi(m)$$

$$\langle\phi_n|\mathcal{W}|\phi_n\rangle = n + i\beta \Leftrightarrow \langle n|W_\lambda|n\rangle = n + \lambda \tag{28}$$

where $\mathcal{W} = \{Q, r\bar{\psi}_\theta\}$ and where the $|\phi_n\rangle$ states are the states on which the topological observables effectively act, and "no-room" corresponds to nonpreserving winding number potentials which are not incorporated into the present gauge model. All necessary conditions to generate well-defined topological invariants have been taken in both models. We thus have the following identifications:

$$\left(\frac{1-i}{2}\right)\mathcal{W} + hc \equiv \text{Diag } W_\beta$$

$$\text{No room} \equiv \text{Nondiag } W_\beta \tag{29}$$

where Diag corresponds to the diagonal part of the operator. The identification of the invariant $\mathcal{W}$ and Diag W_β operators means that both operators are different representations of the same object. The representation spaces are spanned respectively by the states $|\phi_n\rangle$ and $|n\rangle$. The fact that there is no room for the nondiagonal part of W_β or of $\Pi(m)$ in the gauge approach is due to the absence in the model of winding-number-changing amplitudes. In other words, the gauge model based on the simple action of equation (1) is comparable to our model in its simplest version of simple non-self-interacting loops on the background of an external field to which they couple through the general linear coupling allowed, as the function $g(n)$ in (10) may have the general form $g(n) = \beta n + \lambda$,

$$\beta W + \lambda \tag{30}$$

in which one puts $\beta = \lambda$. This suggests that the gauge model needs nontrivial generalizations to incorporate the nondiagonal parts. The above comparative study shows that such a generalization is necessary and also gives insight on the way to do it. The generalization of the action in equation (1) to include winding-number-changing amplitudes is considered and solved in a parallel study to appear in a forthcoming paper (Mekhfi, 1995).

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REFERENCES

Akhoury, R., and Comtet, A. (1984). *Nuclear Physics B*, **246**, 253.
Alvarez Gaumé, L. (1983). *Communications in Mathematical Physics*, **90**, 161.
Alvez, N. A., Aratyn, H., and Zimmerman, A. H. (1985). *Physical Review D*, **31**, 3298.
Balachandran, A. P., Marmo, G., Skagerstam, B. S., and Stern, A. (1991). *Classical Topology and Quantum States*, World Scientific, Singapore.
Baulieu, L., and Arago de Carvalho, C. (1991a). *Physics Letters B*, **275**, 323.
Baulieu, L., and Arago de Carvalho, C. (1991b). *Physics Letters B*, **275**, 335.
Baulieu, L. and Rabinovici, E. (1993). *Physics Letters B*, **316**, 93.
Birmingham, D., and Rakowski, M. (1989). *Modern Physics Letters A*, **4**, 1753.
Birmingham, D., Rakowski, M., and Thompson, G. (1990). *Nuclear Physics B*, **329**, 83.
Birmingham, D., Blau, M., Rakowski, M., and Thompson, G. (1991). *Physics Reports*, **209**, 129.
De Alfaro, V., Fubini, S., and Furlan, G. (1976). *Nuovo Cimento*, **34A**, 569.
Delduc, F., Gieres, F., and Sorella, S. P. (1989). *Physics Letters B*, **225**, 367.
Friedan, D., and Windey, P. (1984). *Nuclear Physics B*, **235**, 395.
Fubini, S., and Rabinovici, E. (1984). *Nuclear Physics B*, **245**, 17.
Hu, S. T. (1959). *Homotopy Theory*, Academic Press, New York.
Maunder, C. R. F. (1979). *Algebraic Topology*, Van Nostrand, New York.
Myers, R. (1990). *International Journal of Modern Physics A*, **5**, 1369.
Mekhfi, M. (1995). Cohomological Quantum Mechanics and Calculability of Observables to appear in IJMPA.
Mekhfi, M. (n.d.). Topological setting of Bessel functions, ICTP preprint, to appear.
Witten, E. (1982). *Nuclear Physics B*, **202**, 253.
Witten, E. (1988a). *Communications in Mathematical Physics*, **117**, 353.
Witten, E. (1988b). *Communications in Mathematical Physics*, **118**, 601.
Witten, E. (1988c). *Physics Letters B*, **206**, 601.